Planar graphs with maximum degree $\Delta \geq 9$ are $(\Delta + 1)$-edge-choosable.

A short proof

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Abstract

We give a short proof of the following theorem due to Borodin [2]. Every planar graph $G$ with maximum degree at least 9 is $(\Delta(G) + 1)$-edge-choosable.

Keywords: edge-colouring, list-colouring, planar graph, List Colouring Conjecture

1. Introduction

All graphs considered in this paper are simple and finite. A proper edge-colouring of a graph $G$ is a mapping $\phi$ from $E(G)$ into a set $S$ of colours such that incident edges have different colours. If $|S| = k$, then $f$ is a proper $k$-edge-colouring. A graph is $k$-edge-colourable if it has a proper $k$-edge-colouring. The chromatic index $\chi'(G)$ of a graph $G$ is the least $k$ such that $G$ is $k$-edge-colourable.

Since edges sharing an endvertex need different colours, $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of $G$. The celebrated Vizing’s Theorem [13] (also shown independently by Gupta [5]) states that $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.

Theorem 1 (Vizing [13]). If $G$ is a graph then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

An edge-list-assignment of a graph $G$ is a function $L$ that assigns to each edge $e \in E(G)$ a list of colours $L(e)$. An edge-list-assignment is $k$-uniform if each list is of size at least $k$. An $L$-edge-colouring of $G$ is a proper edge-colouring $f$ such that $\forall v \in V(G), f(v) \in L(v)$. A graph $G$ is $L$-edge-colourable if there exists an $L$-edge-colouring of $G$. It is $k$-edge-choosable if it is $L$-colourable for every $k$-uniform edge-list-assignment $L$. The choice index or list chromatic index $ch'(G)$ is the least $k$ such that $G$ is $k$-edge-choosable.

One of the most celebrated conjectures on graph colouring is the List Colouring Conjecture asserting that the chromatic index always equals the list chromatic index.

Conjecture 2 (List Colouring Conjecture). For every graph $G$, $\chi'(G) = ch'(G)$.
Bollobás and Harris [1] proved that \( ch'(G) < c\Delta(G) \) when \( c > 11/6 \) for sufficiently large \( \Delta \). Using probabilistic methods, Kahn [9] proved Conjecture 2 asymptotically: \( ch'(G) \leq (1 + o(1))\Delta(G) \). The error term was sharpened by Häggkvist and Janssen [7]: \( ch'(G) \leq \Delta(G) + O(\Delta(G)^{2/3}\sqrt{\log \Delta(G)}) \) and later by Molloy and Reed [10]: \( ch'(G) \leq \Delta(G) + O(\Delta(G)^{1/2}(\log \Delta(G))^4) \).

Galvin [6] proved the List Colouring Conjecture for bipartite graphs. (See also Slivnik [12]).

The List Colouring Conjecture and Vizing’s Theorem imply the following conjecture:

**Conjecture 3.** For any graph \( G \), \( ch'(G) \leq \Delta(G) + 1 \).

This conjecture holds easily when \( \Delta(G) \leq 2 \). It is also true when \( \Delta(G) = 3 \) as in this case the line-graph \( L(G) \) of \( G \) has maximum degree 4 and so \( ch'(G) = ch(L(G)) \leq 5 \) as shown by Juvan et al. [8]. Borodin [2] settled this conjecture for planar graphs of maximum degree at least 9.

**Theorem 4** (Borodin [2]). If \( G \) is a planar graph of maximum degree at least 9, then \( G \) is \( (\Delta(G) + 1) \)-edge-choosable.

This theorem does not imply the List Colouring Conjecture for planar graphs of large maximum degree. Indeed, Sanders and Zhao [11] showed that a planar graph \( G \) with maximum degree at least 7 is \( \Delta(G) \)-edge-colourable. Vizing’s Edge-Colouring Conjecture [14] asserts that \( \Delta(G) \)-edge-colourability also holds for planar graphs with maximum degree 6. Proving this for \( \Delta(G) = 6 \) would be best possible as for \( k \in \{2, 3, 4, 5\} \), there are some planar graphs with maximum degree \( k \) and chromatic index equal to \( k + 1 \) [14].

Borodin, Kostochka and Woodall [3] showed that if \( G \) is planar and \( \Delta(G) \geq 12 \), then \( ch'(G) \leq \Delta(G) \), thus proving the List Colouring Conjecture for planar graphs with maximum degree at least 12. Another proof was given by Cole, Kowalik and Škrekovski [4]; it yields a linear-time algorithm to \( L \)-edge-colour a planar graph \( G \) for any \( \max \{\Delta(G), 12\} \)-uniform edge-list-assignment. Conjecture 3 is still open for planar graphs with maximum degree between 5 and 8, and it is still unknown whether every planar graph with maximum degree between 6 and 11 is \( \Delta(G) \)-edge-choosable.

In this paper, we give a short proof of Theorem 4.

### 2. Proof of Theorem 4

Our proof uses the discharging method.

A vertex of degree \( d \) (respectively at least \( d \), respectively at most \( d \)) is said to be a \( d \)-vertex (respectively a \( d^+ \)-vertex, respectively a \( d^- \)-vertex). The notion of a \( d \)-face (respectively a \( d^+ \)-face, respectively a \( d^- \)-face) is defined analogously regarding the length of a face.

Consider a minimal counterexample \( G \) to the theorem. Let \( L \) be a \( (\Delta(G) + 1) \)-uniform ledge-list-assignment such that \( G \) is not \( L \)-edge-colourable. The graph \( G \) has no edge \( uv \) such that \( d(u) + d(v) \leq \Delta(G) + 2 \), since otherwise any \( L \)-colouring of \( G \setminus uv \) could be extended to one of \( G \) by giving to \( uv \) a colour distinct from the colours of its at most \( \Delta(G) \) adjacent edges. In particular, \( \delta(G) \geq 3 \), and for \( i \geq 3 \) the neighbours of an \( i \)-vertex have degree at least \( \Delta(G) + 3 - i \).

For each \( i \), let \( V_i \) be the set of \( i \)-vertices.
Claim 1. $|V_{\Delta(G)}| > 2|V_3|$. 

Proof. Let $F$ the set of edges in $G$ having one endvertex of degree 3 (hence the other endvertex of degree $\Delta(G)$). Let $H$ be the bipartite subgraph with vertex set $V_3 \cup V_{\Delta(G)}$ and edge set $F$.

We show first that $H$ is a forest. Suppose by way of contradiction that $H$ has a cycle $C$. Since $H$ is bipartite, $C$ is even. By minimality of $G$, $G \setminus E(C)$ has an $L$-edge-colouring. Now every edge of $C$ has at least two available colours since it is incident to $\Delta(G) + 1$ edges, of which $\Delta(G) - 1$ are coloured. Since even cycles are 2-edge-choosable, one can extend the $L$-edge-colouring to $G$, which is a contradiction.

Now, since every vertex of $V_3$ has degree 3 in $H$, we conclude that $|E(H)| = 3|V_3|$, and hence $|V_{\Delta(G)}| + |V_3| > 3|V_3|$. \hfill \qed

Let us assign to each vertex or face a charge equal to its degree (or length) minus 4. It follows easily from Euler’s Formula that $\sum_{x \in V(G)}(d(x) - 4) + \sum_{x \in F(G)}(d(x) - 4) = -8$. Let us now discharge the following rules:

(R1) Every $\Delta(G)$-vertex gives 1/2 to a common pot from which each 3-vertex receives 1;

(R2) Every 8$^+$-vertex gives 1/2 to each of its incident 3-faces;

(R3) Every $d$-vertex with $d \in \{5, 6, 7\}$ gives $\frac{d-4}{d}$ to each of its incident 3-faces.

We show that the final charge $f(x)$ for every vertex or face is nonnegative. We also show that the final charge of the common pot is nonnegative. This implies that the total final charge is nonnegative; since the total final charge equals the total initial charge, this is a contradiction.

\begin{itemize}
\item Since $|V_{\Delta(G)}| > 2|V_3|$ by Claim 1, the charge of the common pot is positive.
\item Let $x$ be a $d$-vertex.
\end{itemize}

If $d = 3$, then $x$ receives 1 from the pot and gives no charge away, so $f(x) \geq 0$. If $d = 4$, the charge of $x$ does not change, so $f(x) = d - 4 = 0$. If $d \in \{5, 6, 7\}$, then $x$ sends at most $\frac{d-4}{d}$ to each of its incident face so $f(x) \geq d(1 - \frac{d-4}{d}) - 4 \geq 0$. If $8 \leq d \leq \Delta(G) - 1$, then $x$ sends at most 1/2 to each of its incident faces, so $f(x) \geq d - 4 - d/2 \geq 0$. If $d = \Delta(G)$, then $x$ loses charge 1/2 to the pot and 1/2 to each incident 3-face, so $f(x) \geq d - 4 - d/2 - 1/2 \geq 0$, since $d \geq 9$.

\begin{itemize}
\item Let $x$ be a $d$-face.
\end{itemize}

If $d \geq 4$, then its charge does not change so $f(x) = d(x) - 4 \geq 0$. Suppose now that $d = 3$. If $x$ contains a 4$^-$-vertex, then the two other neighbours have degree at least $\Delta(G) - 1$, so $x$ receives 1/2 from each of them. Thus $f(x) = 3 - 4 + 2 \times 1/2 = 0$. If $x$ contains a 5-vertex then its two other vertices have degree at least $\Delta(G) - 2$ which is at least 7. Hence $x$ receives at least $\frac{1}{2}$ from its 5-vertex and at least $\frac{3}{7}$ from the other two vertices. So $f(x) \geq 3 - 4 + 1/5 + 2 \times 3/7 > 0$. Otherwise, all the vertices incident to $x$ are 6$^+$-vertices. Hence $f(x) \geq 3 - 4 + 3 \times 1/3 = 0$.

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